

Pseudo-Gauss distribution: a reversible counterpart of the true turbulent velocity gradient distribution

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Abstract

On the grounds of both widely known experimental and numerical data of the strain-rate tensor statistical properties in the fully developed incompressible turbulent flow and the integral transformations deduced in the article, some important statistical properties of the flow, known as kinematical, are found to be dynamical (i.e. taking place purely in a turbulent flow). A new pseudo-Gauss distribution is introduced in the article and, as a consequence of these properties, a considerable feature distinguishing this distribution from the Gauss one is found to be the general feature of the turbulent statistics. Thus, the new distribution is proved to be a reversible counterpart of the true turbulent velocity gradient distribution instead of the Gauss one.

1.Introduction

The velocity field in the fully developed turbulent flow is known to be stochastic. Its statistical properties have attracted the interest of researchers in physics, mechanics and mathematics for more than a century, but the problem is still far from being solved. Previously, a new model [1], which gives a physical interpretation of the known multifractal model, was formulated. The large-scale velocity gradients play a significant role in this model. Now, we try to focus our interest on the statistical properties of the velocity gradients in the fully-developed turbulent flow.

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Full velocity gradient statistical description

Velocity gradients compose the tensor, which can be defined as value A_{ij} in a given point \vec{x}_0 from the following expansion of the velocity field in \vec{x}_0 neighborhood:

$$u_i(\vec{x}) = u_i(\vec{x}_0) + A_{ij}(\vec{x}_0)(x_j - x_{0j}) + O(|\vec{x} - \vec{x}_0|^2) \quad (1)$$

This tensor is called the velocity gradient tensor and as it is evident from (1):

$$A_{ij}(\vec{x}_0) = \frac{\partial u_i}{\partial x_j} \Big|_{\vec{x}=\vec{x}_0} \quad (2)$$

It follows from the incompressibility of the flow that the trace of the velocity gradient tensor is equal to zero:

$$A_{ii} = 0 \quad (3)$$

Under these conditions, the velocity gradient tensor has 8 independent components. It has 5 rotational invariants which are independent of the frame orientation. For instance, the invariants can be chosen as [2]:

$$Q = -\frac{1}{2}A_{im}A_{mi} \quad R = -\frac{1}{3}A_{im}A_{mn}A_{ni} \quad (4)$$

$$Q_S = -\frac{1}{2}S_{im}S_{mi} \quad R_S = -\frac{1}{3}S_{im}S_{mn}S_{ni} \quad (5)$$

$$V^2 = S_{in}S_{im}\omega_m\omega_n \quad (6)$$

Here S_{ij} is a strain-rate tensor and $\vec{\omega}$ - vorticity of the flow. They are both components of the velocity tensor and can be defined by decomposition:

$$A_{ij} = \frac{1}{2}(A_{ij} + A_{ji}) + \frac{1}{2}(A_{ij} - A_{ji}) = S_{ij} - \frac{1}{2}\epsilon_{ijm}\omega_m \quad (7)$$

Here ϵ_{ijm} is the antisymmetric tensor. Because of $SO(3)$ symmetry the probability density functions (PDFs) of velocity gradients depend on rotational invariants only [3]. It could be expressed in terms of invariants (4)-(6) or in any other five independent invariants, which are functions of invariants (4)-(6).

Symmetric velocity gradients statistical description

This article focuses on the statistical properties of the symmetrical part of decomposition (7). One can see from (5) that invariants Q_S and R_S are functions of strain-rate tensor S_{ij} only. Due to the incompressibility and symmetrical arguments S_{ij} has only two invariants which are independent of the frame orientation. Thus, the statistical properties of strain-rate tensor S_{ij} can be considered independently in terms of Q_S and R_S invariants or in other two independent invariants, which must be functions of Q_S and R_S . For instance, new invariants, namely, the maximum (λ_1) and the minimum (λ_3) eigenvalues of strain-rate tensor S_{ij} could be chosen instead of Q_S and R_S (due to the incompressibility, the third eigenvalue λ_2 , which is greater than λ_3 but less than λ_1 , is equal to $-\lambda_1 - \lambda_3$). The expressions of Q_S and R_S in terms of λ_1 and λ_3 take the form:

$$Q_S = -(\lambda_1^2 + \lambda_1\lambda_3 + \lambda_3^2) \quad (8)$$

$$R_S = \lambda_1\lambda_3(\lambda_1 + \lambda_3) \quad (9)$$

The inverse transform to (8), (9) can be shown to be unique, but much more complicated.

There are other parameters, single PDFs of which do not give the full statistic information of the strain but are useful in considering of fluid particle deformation tendencies.

$$\beta = \frac{\sqrt{6}\lambda_2}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}} \quad (10)$$

$$s = \frac{-3\sqrt{6}\lambda_1\lambda_2\lambda_3}{(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)^{\frac{3}{2}}} \quad (11)$$

They were introduced and analyzed in articles [4, 5] respectively. In an incompressible flow both of them can change in the interval $[-1; 1]$. Each of these values gives a unique value of eigenvalues ratio $\lambda_1 : \lambda_2 : \lambda_3$ but says nothing about their absolute values. The expression coupling two parameters β and s has been given in the article [5], so they cannot be chosen as two independent parameters of S_{ij} and, thus, it is surely senseless to find the joint PDF of the parameters.

Physical interpretation of velocity gradients

Note that defined tensors A_{ij} and S_{ij} , vorticity vector $\vec{\omega}$ and the S_{ij} eigen values (with the corresponding eigen vectors) have a physical interpretation in terms of fluid particles - macroscopic but small volumes of the moving flow. Let \vec{x}_0 in (2) be the position vector of the fluid particle center of mass. If the fluid particle dimensions are small enough to apply the linear expansion (1), velocity gradient tensor A_{ij} entirely governs its dynamics in its center-of-mass system in a given moment: the rotation and the deformation of the particle are governed by vorticity vector $\vec{\omega}$ and strain-rate tensor S_{ij} accordingly. As for the deformation of the fluid particle, three principal axes of the strain can be distinguished in the directions of S_{ij} eigen vectors with the exponential rate of strain in these directions. Namely, the expansion with relative rate λ_1 occurs in the direction of the eigen vector corresponding to these directions, the contraction with rate $|\lambda_3|$ in its eigen vector direction and either the expansion or the contraction (depending on the sign of λ_2) with rate $|\lambda_2|$ take place in the corresponding eigen vector direction [6]. Hence, the ratio $\lambda_1 : \lambda_2 : \lambda_3$ (either parameter β or s) characterizes the shape of the ellipsoidal deformation of an initially spherical fluid particle. For example, if $\beta = +1$ (or $s = +1$) then $\lambda_1 : \lambda_2 : \lambda_3 = 1 : 1 : -2$ and the initially spherical fluid particle becomes a pancake-like. If $\beta = -1$ (or $s = -1$) then $\lambda_1 : \lambda_2 : \lambda_3 = 2 : -1 : -1$ and it tends to be a filament.

Generalization of velocity gradients on large scales

If we consider so large fluid particles that the expansion (1) becomes wrong, the velocity gradient tensor, defined in (2), fails to govern their dynamics. However, it is those particles, which are known to be of huge interest in the turbulence theory [7]. Therefore, the definition of the velocity gradient tensor (in [8] it is called "true" velocity gradient tensor) should be generalized on different scales, for example, with the help of the following averaging:

$$A_{ij}^R(\vec{x}_0) = \frac{1}{V_R} \int_{V_R} A_{ij}(\vec{x}_0 + \vec{r}) d\vec{r} \quad (12)$$

Where V^R is the spherical volume with the center in the point \vec{x}_0 , which radius is equal to R . It is possible to split generalizing tensor A_{ij}^R into symmetrical and antisymmetric parts S_{ij}^R and $-\frac{1}{2}\epsilon_{ijm}\omega_m^R$ analogous to A_{ij} (7). The traces of tensors A_{ij}^R and S_{ij}^R are obviously equal to zero, so it has five and two nonzero orientation invariants respectively, which are defined in the same way as it was done previously. Therefore, general statistical analysis is analogous for both true and generalized gradient tensors. Due to this, the results deduced in the article are valid for both tensors.

Short overview of the previous results

All above discussed quantities were calculated in many direct numerical simulations (DNS) and were measured in a wide range of experiments (see the overview [9] and the references in it). There are also many phenomenological theoretical models for these quantities (see the overview [2] and the references in it). Note that most of numerical, experimental and theoretical articles on the velocity gradients deal with joint PDF of Q and R velocity gradient tensor invariants ([2, 9] and references).

Concerning strain-rate tensor invariants, note that DNS are usually focused on the joint PDF of invariants Q_S and R_S (e.g. [10, 11]) and, on the contrary, in experiments researchers usually prefer to measure single distributions of λ_1 , λ_2 and λ_3 (e.g. [12, 13])¹. Single β and s PDFs were measured and calculated in a considerable amount of various fully developed incompressible and slightly compressible flows. Their properties has been investigated since they were calculated with low Reynolds numbers (β [4, 5, 14, 15], s [5]) and measured by hot-wire probes (β [16]). There are modern precise calculations (β [8, 17], s [17]) and measurements by both hot-wire probes (β and s [17]) and PIV methods (β [8]). The important property of β PDF is that it vanishes at the end points of its domain (i.e. at points $\beta = \pm 1$). It can be shown [5] that this means s PDF boundedness at $s = \pm 1$, which are end points of its domain.

Present article results

In contradiction to the known results of the article [5]², we will show that the properties of single β and s PDFs give non-trivial information about turbulent statistics (see section 4). Before this, in section 3, we will show that in case of the Gauss velocity distribution these properties do not take place (i.e. β PDF does not vanish at $\beta = \pm 1$ and s PDF tends to infinity in $s = \pm 1$) and will give the reversibility analog of the Gauss distribution that we have called the pseudo-Gauss distribution, for which these properties take place. All the results will be found by means of the integral transformations between different PDFs that we will deduce in section 2. Note that it is convenient to give the resulting property of turbulent statistics not only in terms of invariants Q_S and R_S but also via "shuffled" eigenvalues of the strain-rate tensor and their joint PDF, which we will introduce in section 2.

In our opinion, the contradiction of our results with the previous ones is associated

¹Sometimes single distributions of Q_S [11] and R_S [12, 13] can be found both in numerical and experimental works.

²And quite many subsequent articles that refer to its conclusions (e.g. overview [2], interpretation of experimental [18, 19, 20, 21] and DNS [20, 21] results, phenomenological model [22] and development of the ideas of [5] on the compressible flows [23])

with our account of pre-exponential factor in the joint PDF of invariants Q_S and R_S in Gauss case, which has not been accounted before (see section 3).

2. Joint and single PDFs of different strain-rate tensor invariants and relations between them

"Shuffled" and "ordered" eigen values

Let λ_1 , λ_2 and λ_3 be eigen values of the strain-rate tensor S_{ij} ordered by the condition $\lambda_1 \geq \lambda_2 \geq \lambda_3$ as they are defined in the introduction. Let us introduce the procedure, which we call "shuffling". This means that we set the "ordered" eigenvalues λ_1 , λ_2 and λ_3 into one-to-one correspondence with so-called "shuffled" eigenvalues μ_1 , μ_2 and μ_3 letting each be equal to one of three λ_i with the same probability $\frac{1}{3}$. Due to the one-to-one correspondence, it is evident that the shuffled eigenvalues hold the incompressibility condition $\mu_1 + \mu_2 + \mu_3 = 0$, too. Moreover, two of the shuffled eigen values (e.g. μ_1 and μ_2) can be marked as new strain-rate tensor invariants like Q_S and R_S or λ_1 and λ_3 . Thus, as we have mentioned in introduction, PDFs of the functions depending on the strain-rate tensor can be expressed in terms of shuffled eigen values.

Joint PDFs of the shuffled eigen values (μ PDFs)

Let us denote the joint PDF of the shuffled eigen values μ_i and μ_j ($i \neq j$) $f_{ij}(x, y)$. By the construction of the procedure we have 4 simple properties for μ PDFs:

- 1) Domain $[f_{ij}(x, y)] = \mathbb{R}^2$
- 2) $f_{ij}(x, y) = f_{ji}(x, y)$
- 3) $f_{12}(x, y) = f_{23}(x, y) = f_{13}(x, y)$
- 4) $f_{ij}(x, y) = f_{ij}(y, x) = f_{ij}(x, -x - y)$

These properties make "shuffled" eigenvalues more convenient than the ordered ones and we will use them and their joint PDF hereafter³. The inverse relations between the shuffled eigen values and the ordered ones are explicit and are given by the formulas:

$$\lambda_1 = \max(\mu_1, \mu_2, -\mu_1 - \mu_2) \quad (13)$$

$$\lambda_3 = \min(\mu_1, \mu_2, -\mu_1 - \mu_2) \quad (14)$$

Thus, the given procedure is reversible. The four properties of the μ PDFs can be taken into account by only considering the following functions:

$$f_{ij}(x, y) = \phi(-(x^2 + xy + y^2), xy(x + y)) \quad (15)$$

where $(x; y) \in \mathbb{R}^2$.

³The more detailed discussion of the ordered eigenvalues and their joint PDF see in Appendix 1.

Joint Q_S and R_S PDF

Let us find the relation between introduced PDF $f_{ij}(x; y)$ and joint Q_S and R_S PDF $f_{Q_S R_S}$. Invariants Q_S and R_S are expressed by μ_i and μ_j for any i and j ($i \neq j$) in the following way:

$$\begin{cases} Q_S = -(\mu_i^2 + \mu_i \mu_j + \mu_j^2) \\ R_S = \mu_i \mu_j (\mu_i + \mu_j) \end{cases} \quad (16)$$

One can show that exactly six pairs of μ_i and μ_j correspond to one pair of values Q_S and R_S .⁴ This divides the space $(\mu_i; \mu_j)$ in 6 independent sectors, separated from each other by the lines on which two of the three eigenvalues are equal:

$$\begin{cases} \mu_i = \mu_j \\ \mu_i = -\mu_i - \mu_j \\ \mu_j = -\mu_i - \mu_j \end{cases} \quad (17)$$

The domain of Q_S and R_S ($D_{Q_S R_S}$), which is shown in Fig.1, is bounded by the curves $R_S = \pm \frac{2}{3\sqrt{3}} (-Q_S)^{\frac{3}{2}}$ in the lower half plane [10, 11]. To find the relation between the introduced joint PDF and the joint PDF of Q_S and R_S invariants the Jacobian of the transformation (16) is needed. Then:

$$f_{Q_S R_S}(q; r) = 6 \cdot \phi(q; r) \left\| \frac{\partial(\mu_i, \mu_j)}{\partial(Q_S, R_S)} \right\|_{Q_S=q; R_S=r} = \frac{6 \cdot \phi(q; r)}{\sqrt{-4q^3 - 27r^2}} \quad (18)$$

Where 6 is the number of μ_i and μ_j pairs, which correspond to the pair of values Q_S and R_S from (16). Also note that $(Q_S; R_S) \in D_{Q_S R_S}$.

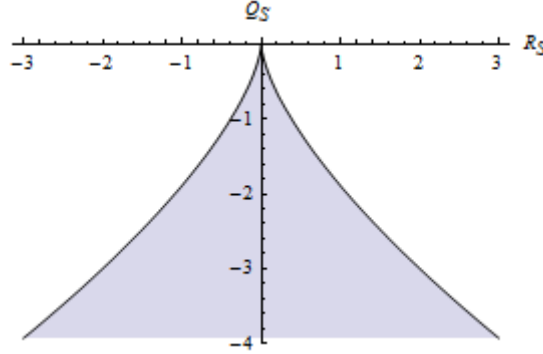


Figure 1: $D_{Q_S R_S}$: the domain of Q_S and R_S invariants

Hence, knowing μ PDF we can find the Q_S and R_S PDF by applying (18). From (16) and (18) we get the inverse relation:

$$\begin{aligned} f_{ij}(x; y) &= \\ &= \frac{|(x-y)(2x+y)(x+2y)|}{6} f_{Q_S R_S}(- (x^2 + xy + y^2), xy(x+y)) \end{aligned} \quad (19)$$

⁴As transform (8), (9) is unique, nonuniqueness of reverse transformation, in fact, is associated with "shuffling"

where $(x; y) \in \mathbb{R}^2$.

Single s PDF

Using (11), it is possible to get s PDF f_s from $f_{Q_S R_S}$. It can be done by the integral transformation:

$$f_s(y) = \int_{-\infty}^0 dq \int_{-\frac{2}{3\sqrt{3}}(-q)^{\frac{3}{2}}}^{\frac{2}{3\sqrt{3}}(-q)^{\frac{3}{2}}} f_{Q_S R_S}(q; r) \delta\left(y - \frac{3\sqrt{3}r}{2(-q)^{\frac{3}{2}}}\right) dr \quad (20)$$

Here $\delta(y - g(q; r))$ is Dirac delta function on the surface $g(q; r)$ [24] convoluting with $f_{Q_S R_S}$ in its domain $D_{Q_S R_S}$, which is shown in Fig. 1. The calculation (20) could be reduced to the following expression:

$$f_s(y) = \frac{4}{3\sqrt{3}} \int_0^{+\infty} f_{Q_S R_S}\left(-t^2; t^3 \frac{2y}{3\sqrt{3}}\right) t^4 dt \quad (21)$$

where $s \in [-1; 1]$. From the transformation (21) and (18) the similar transformation between the s PDF and the μ PDF could be determined:

$$f_s(y) = \frac{4}{\sqrt{3}\sqrt{1-y^2}} \int_0^{+\infty} \phi\left(-t^2; t^3 \frac{2y}{3\sqrt{3}}\right) t dt \quad (22)$$

Single β PDF

Now, let us find analogous integral transformations for parameter β . To do it, let us use the explicit relations between parameters β and s given in [5]:

$$\begin{cases} \beta = 2 \sin\left(\frac{1}{3} \arcsin s\right) \\ s = \beta (3 - \beta^2) / 2 \end{cases} \quad (23)$$

Determining the Jacobian of transformation (23) and using (21) we can find:

$$\begin{aligned} f_\beta(z) &= f_s\left(\frac{z(3-z^2)}{2}\right) \left| \frac{ds}{d\beta} \right|_{\beta=z} = \\ &= \frac{2}{\sqrt{3}} (1-z^2) \int_0^{+\infty} f_{Q_S R_S}\left(-t^2; t^3 \frac{z(3-z^2)}{3\sqrt{3}}\right) t^4 dt \end{aligned} \quad (24)$$

where $z \in [-1; 1]$. Analogously we can determine the integral transformation from joint PDF ϕ to PDF f_β :

$$f_\beta(z) = \frac{4\sqrt{3}}{\sqrt{4-z^2}} \int_0^{+\infty} \phi\left(-t^2; t^3 \frac{z(3-z^2)}{3\sqrt{3}}\right) t dt \quad (25)$$

3. Gauss and pseudo-Gauss distribution

Gauss distribution case

Let us consider the random velocity field with Gauss statistics. In the article ([25] Appendix B) it was shown that from the Gaussian distribution of the flow

velocity follows the Gaussian distribution of the velocity gradient tensor elements. From the form given in [25] one can write for the strain-rate tensor eigen values PDF:

$$f_{ij}(x; y) = \phi(-(x^2 + xy + y^2)) = \frac{\sqrt{3}}{2\pi M^2} \exp\left(\frac{-x^2 - xy - y^2}{M^2}\right) \quad (26)$$

Here M is a parameter characterizing the distribution dispersion, which in the incompressible case is governed only by the mean value of squared vorticity ω^2 [25]. To find Q_S and R_S joint PDF let us take into account (18):

$$f_{Q_S R_S}(q; r) = \frac{3\sqrt{3} \exp(q/M^2)}{\pi M^2 \sqrt{-4q^3 - 27r^2}} \quad (27)$$

Substitution (26) in (25) and (22) gives β and s PDFs for the Gauss velocity distribution:

$$f_\beta(z) = \frac{3}{\pi \sqrt{4 - z^2}} \quad (28)$$

$$f_s(y) = \frac{1}{\pi \sqrt{1 - y^2}} \quad (29)$$

These PDFs are shown in Fig. 2. We see that in this case f_β does not vanish at the points ± 1 as it was concluded in [5] (see figure 1 from [5]). Moreover, in contrast with [5] f_s becomes infinity in the points ± 1 . In our opinion, this contradiction is associated with our account of pre-exponential factor $(-4q^3 - 27r^2)^{-1/2}$ in (27), which was not accounted before. To prove this statement we now consider Q_S and R_S joint PDF without the pre-exponential factor.

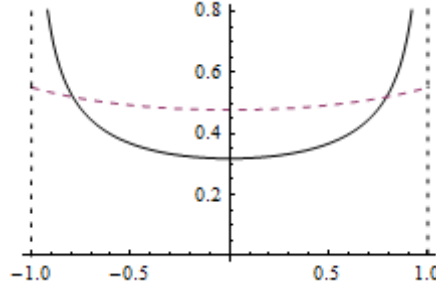


Figure 2: PDFs f_s (the full curve with the vertical asymptotes) and f_β (the dotted curve) in case of Gauss velocity field.

Pseudo-Gauss distribution case

Now let us consider the case of symmetric exponential distribution $f_{Q_S R_S}$ (without the pre-exponential factor) that we have called a pseudo-Gauss distribution:

$$f_{Q_S R_S}(q; r) = \frac{\sqrt{3} \exp(q/A^2)}{\sqrt{\pi} A^5} \quad (30)$$

In contrast to M from (26), parameter A still has no intuitive physically meaningful interpretation because we do not have the pseudo-Gauss distribution as a consequence of any specific velocity field like in the previous subsection. The

inverse problem of finding the velocity distribution from distribution (30) is not considered in the present article and the question still remains open.

By substituting (30) in (19) we have:

$$f_{ij}(x; y) = \frac{\sqrt{3} |(x-y)(2x+y)(2y+x)|}{6\sqrt{\pi}A^5} \exp\left(\frac{-x^2 - xy - y^2}{A^5}\right) \quad (31)$$

One can see that this f_{ij} has a very specific shape as it vanishes on the lines $x = y$, $x = -x - y$ and $y = -x - y$, that is, when two of three strain-rate eigenvalues are equal $\beta = \pm 1$ (or $s = \pm 1$). Using (21), (24) and (30) after some developments, we have:

$$f_{\beta}(z) = \frac{3}{4} (1 - z^2) \quad (32)$$

$$f_s(y) = \frac{1}{2} \quad (33)$$

Derived PDFs f_{β} and f_s (Fig. 3) are exactly the same as the ones in figure 1 in the article [5].

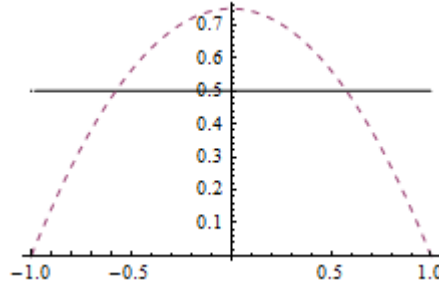


Figure 3: PDFs f_s (the full curve) and f_{β} (the dotted one) in case of pseudo-Gauss velocity field.

Q-distribution cases

Let us notice that there is no parameter M in expressions (28) and (29), so, f_{β} and f_s are independent of the dispersion in (26). We now show that, moreover, the distributions (28) and (29) are valid not only for the Gauss distribution case, but for all symmetric PDFs f_{ij} .

Statement 1. If $\phi(q; r) = \phi(q)$, then $f_s(y) = 1/\pi\sqrt{1-y^2}$ and $f_{\beta}(z) = 3/\pi\sqrt{4-z^2}$

Indeed, it follows from (22):

$$f_s(y) = \frac{4}{\sqrt{3(1-y^2)}} \int_0^{+\infty} \phi(-t^2) t dt \quad (34)$$

Since the integral from the right side does not depend on y and PDF $f_s(y)$ is normalized, the integral must fail to be governed by function $\phi(q)$:

$$\int_0^{+\infty} \phi(-t^2) t dt = \frac{\sqrt{3}}{4} \left(\int_{-1}^1 \frac{dy}{\sqrt{1-y^2}} \right)^{-1} = \frac{\sqrt{3}}{4\pi} \quad (35)$$

This leads to the equation for $f_s(y)$. In the same manner, from (25) we easily can prove the one for $f_{\beta}(z)$:

$$f_{\beta}(z) = \frac{4\sqrt{3}}{\sqrt{4-z^2}} \int_0^{+\infty} \phi(-t^2) t dt = \frac{3}{\pi\sqrt{4-z^2}} \quad (36)$$

The analogous notation can be made about parameter A in (30) and distributions (32) and (33), which are independent of it. Let us prove that distributions (32) and (33) take place not only in the pseudo-Gauss case but also for all symmetric PDFs $f_{Q_S R_S}$:

Statement 2. If $f_{Q_S R_S}(q; r) = f_{Q_S R_S}(q)$, then $f_s(y) = 1/2$ and $f_{\beta}(z) = 3/4 \cdot (1 - z^2)$

From (21) we can write:

$$f_s(y) = \frac{4}{3\sqrt{3}} \int_0^{+\infty} f_{Q_S R_S}(-t^2) t^4 dt \quad (37)$$

The same method of f_s normalization provides:

$$\int_0^{+\infty} f_{Q_S R_S}(-t^2) t^4 dt = \frac{3\sqrt{3}}{8} \quad (38)$$

This proves the first part of the statement. Finally, from (24) it turns out:

$$f_{\beta}(z) = \frac{2}{\sqrt{3}} (1 - z^2) \int_0^{+\infty} f_{Q_S R_S}(-t^2) t^4 dt = \frac{3}{4} (1 - z^2) \quad (39)$$

Note that (35) and (38) are deduced by f_s normalization instead of f_{ij} and $f_{Q_S R_S}$ normalization. The direct proof is more complicated and is given in Appendix 2.

4. Zeroes structure of the shuffled eigen values PDF

Let us deduce the properties of the turbulent flow that governs the specific shape of distributions f_{β} and f_s , which can be seen from figure 1 in [5] and written as:

$$f_{\beta}(\pm 1) = 0 \quad f'_{\beta}(\pm 1) \neq 0 \quad f_s(\pm 1) \neq 0 \quad f_s(\pm 1) \neq \infty \quad (40)$$

Not all of these properties are independent. The dependence of the first and the fourth properties is shown in the article [5]. It is easy to show that the second and the third ones are identical, too. In fact, relation (23) enables to represent statistical properties of the flow in terms of f_{β} and f_s , equivalently (the similar fact emerges for f_{ij} and $f_{Q_S R_S}$ expressing each other via (16), (18) and (19)). To formulate one of the properties in a convenient way let us introduce a secondary function g_{ij} by the following expression:

$$f_{ij}(x; y) = |(x - y)(2x + y)(2y + x)| g_{ij}(x; y) \quad (41)$$

Statement 3. If conditions (40) hold, then

- 1) $f_{Q_S R_S}(q; r)$ is limited on the curve $-4q^3 - 27r^2 = 0$, however, it is not identical to be equal to zero on it.
- 2) $g_{ij}(x, y)$ is limited on the lines $x = y$, $x = -2y$ and $2x = -y$, however, it is not identical to be equal to zero on them.

$\phi(q; r)$ can be represented uniquely in the form:

$$\phi(q; r) = (-4q^3 - 27r^2)^\epsilon \gamma(q; r) \quad (42)$$

where $\gamma(q; r)$ is not equal to zero on the curve $-4q^3 - 27r^2 = 0$ and does not tend to infinity on it. By substituting the latter expression in (22) and taking into consideration relations (40) one can get $\epsilon = 1/2$. Applying conditions (18) and (19), we finally prove the statement.

Therefore, shuffled eigenvalues PDF, which has the Gaussian shape in case of Gauss normal velocity distribution, in fact, vanishes on the lines $x = y$, $x = -2y$ and $2x = -y$ corresponding to axisymmetric extension $\beta = +1$ (or $s = +1$) and axisymmetric contraction $\beta = -1$ (or $s = -1$).

5. Conclusions

Comparison of the article results with the previous ones

Let us start our conclusions with considering some important difference between the results of our article and the results of the article [5], which are believed to be right in a wide range of articles following [5] (see the second footnote).

The key distinction is that we deduce PDF f_β of the dimensionless parameter β (10) (in section 3) to be nonzero in the ends of the function domain in the Gauss distribution case. Likewise, PDF f_s of parameter s (11) in our article is deduced (in section 3) to tend to infinity in the ends of its domain, but in [5] it was found to be constant on its domain ($f_s = 1/2$). The fact that f_s is uniform for the Gauss distribution was believed to be the reason for choosing f_s as the most convenient single PDF to illustrate the tendencies of the fluid particles strain in the turbulent flow [2, 5]. Following this logic it should be mentioned that β PDF, which is not constant in the case of the Gauss velocity distribution (see Fig. 2) is more preferable, because it is quite close to be constant (it can possess values from only $f_\beta(0) \simeq 0.48$ to $f_\beta(\pm 1) \simeq 0.55$). As for s PDF, it tends to infinity at the ends of its domain in the Gauss case (see Fig. 2) and, thus, is far from being constant.

From the physical point of view this agrees with the article [4] conclusions that the axisymmetric extension and contraction of fluid particles ($\beta = \pm 1$ or $s = \pm 1$) do not exist in the turbulent flow. It should be mentioned that the absence of $\beta = \pm 1$ ($s = \pm 1$) particles surely does not mean the prevalence of filament structures because the whole interval of values $0 < \beta \leq 1$ leads to the pancake shape and as can be seen from the experimental and numerical results for f_β (e.g. figure 1 in [5]):

$$\int_0^1 f_\beta(z) dz > \int_{-1}^0 f_\beta(z) dz \quad (43)$$

In the present article we found the class of distributions (statement 2 in section 3), for which f_β and f_s have the form that they seemed to have in case of the Gauss distribution (see [5] and the articles from the second footnote on page 3). We called the exponential distribution from this class (30) the pseudo-Gauss distribution and it differs from the Gauss distribution in the absence of the pre-exponential factor $(-4q^3 - 27r^2)^{-1/2}$, which, as we concluded, was not accounted before. This pre-exponential factor leads to the vanish of pseudo-Gauss PDF f_{ij} (31) on the lines corresponding to axisymmetric extension and contraction. Thus, it dramatically differs from the Gauss distribution, which vanishes nowhere. We saw in section 4 that such structure is the result of vanishing f_β at ± 1 and, thus, the latter property is not a trivial one (i.e. kinematical in terms of [5, 25]), but an essential property of the turbulent statistics (i.e. dynamical).

Let us pay attention to the fact that has a physical meaning. An energy cascade from large to small scales is known to arise in the turbulent flow [7] that means macroscopic irreversibility of the turbulent processes. One can conclude that the irreversibility of fluid dynamics appear due to skewness of its PDF as a result of the prevalence of direct processes over the inverse ones. For instance, the Gauss velocity distribution (26) is symmetric and, thus, free of the irreversibility. This is believed to be the keynote distinction between the Gauss and real turbulent velocity distributions. Similarly, via its symmetry, the pseudo-Gauss distribution (31) corresponds to the reversible process and does not contain the irreversible energy cascade. We showed in section 3 that the pseudo-Gauss distribution is qualitatively more suitable for the real turbulent statistics and we generalized this fact in statement 3 (in section 4) on any f_{ij} , deducing f_{ij} to vanish on these lines to satisfy conditions (40). Hence, we find a new ("reversible") reason for the turbulent velocity field nongaussianity.

However, statement 3 is not deduced from the first principles. In fact, it is the consequence of experimental and DNS results (40). Although there are no significant doubts that such a great number of the concordant data gives real information about turbulent statistics, the fundamental reasons and consequences of the unexpected nongaussianity need to be explored in the future.

The first step in this way can be done by finding the example of reversible velocity field statistics, which governs the pseudo-Gauss distribution or any distribution from the class concerned in statement 2. Unfortunately, we have not found it yet.

Joint PDFs of the shuffled eigen values and integral transformations

We think that the PDF of the strain-rate tensor eigen values f_{ij} , which was introduced in section 2, will be convenient for the further research of the tensor statistical properties (especially, in their theoretical investigations) due to its unbounded domain and symmetric properties.

The transformations (18)-(25) that we found in section 2 by means of f_{ij} were useful in deducing the article statements. We believe that these transformations will be useful not only in future theoretical investigations but also for the validation of numerical calculations and experimental results. We also think that the method of finding the exact transformations could give many fruitful results about the velocity gradient tensor, too. In fact, there are many other dimensionless parameters depending also on the vorticity vector (e.g see [2, 8, 9, 21]) and one may hope that deduction of analogous transformations for them could give further nontrivial conclusions about turbulent velocity gradient statistics.

6. Appendix

1. Ordered eigen values and their joint PDF

There are three pairs of independent ordered eigenvalues. They can be chosen as λ_1 and λ_2 ; λ_1 and λ_3 ; λ_2 and λ_3 . Thus, there are six different joint PDFs of ordered eigen values: $f_{\lambda_1\lambda_2}$ and $f_{\lambda_2\lambda_1}$, $f_{\lambda_1\lambda_3}$ and $f_{\lambda_3\lambda_1}$, $f_{\lambda_2\lambda_3}$ and $f_{\lambda_3\lambda_2}$. The functions domains given in Fig. 4 have intersections only on lines $x = y$, $x = -x - y$ and $y = -x - y$, that correspond to axisymmetric extension or contraction.

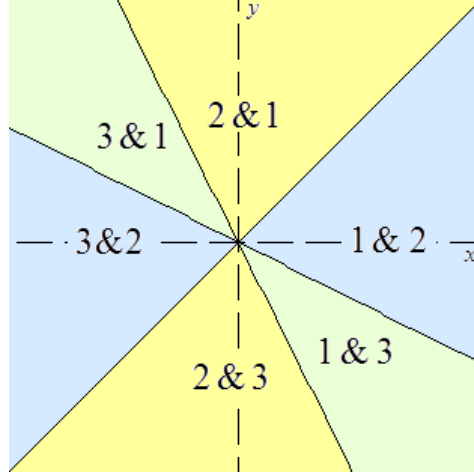


Figure 4: The ordered eigen values PDF domains: $i \& j$ of λ_i and λ_j .

Due to statement 3 it is those lines, on which f_{ij} is equal to zero. The relations between these functions and shuffled eigenvalues PDF are very simple and PDFs of any values λ_i and λ_j ($i \neq j$) can be expressed (in their domain!) by the fully identical formulas:

$$f_{\lambda_i\lambda_j}(x; y) = 6f_{ij}(x; y) \quad (44)$$

Hence, by virtue of statement 3 each function $f_{\lambda_i\lambda_j}(x; y)$ vanishes on the bound of its domain. Despite the simplicity of the latter relation, we consider our PDF of shuffled eigenvalues more useful, especially in theoretical research, since it has an unbounded domain and symmetric properties.

Another deduction of the statement 1.

Since the deductions of statements 1 and 2 are almost similar we give direct deduction of statement 2 only. By condition of statement 2 we have:

$$f_{Q_S R_S}(q; r) = f_{Q_S R_S}(q) \quad (45)$$

It provides from the normalization of $f_{Q_S R_S}$ that:

$$\begin{aligned} 1 &= \iint_{D_{Q_S R_S}} f_{Q_S R_S}(q) dq dr = \\ &= \int_{-\infty}^0 f_{Q_S R_S}(q) dq \int_{-\frac{2}{3\sqrt{3}}(-q)^{3/2}}^{\frac{2}{3\sqrt{3}}(-q)^{3/2}} dr = \frac{4}{3\sqrt{3}} \int_{-\infty}^0 (-q)^{3/2} f_{Q_S R_S}(q) dq \quad (46) \end{aligned}$$

Finally, using (21) we have:

$$f_s(y) = \frac{4}{3\sqrt{3}} \int_0^{+\infty} f_{Q_S R_S}(-t^2) t^4 dt = \frac{4}{3\sqrt{3}} \int_{-\infty}^0 (-q)^2 f_{Q_S R_S}(q) \frac{dq}{2(-q)^1/2} = \frac{1}{2} \quad (47)$$

Statement 1 can be proved identically using (18), (22), (25).

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